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SOLUTION OF GRINBERG AND CHEKMAREVA'S FIRST INTEGRAL EQUATION USING AN ASYMPTOTIC SERIES IN A SMALL PARAMETER THAT IS PRESENT

L. É. Rikenglaz

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A method is suggested for constructing the terms in an asymptotic series in a small parameter μ when seeking the position of the phase front $y(\tau)$ in the Stefan boundary-value problem of the first kind for a semi-infinite medium that is at the phase transition temperature at the initial moment.

1. The nonlinear integral equations obtained in [1] for determining the position of the phase front $\xi(t)$ in Stefan's problem for a semi-infinite body $x \geq 0$ that is at the temperature of the phase transition $T(x, 0) = 0$ at the time $t = 0$ will be referred to as Grinberg and Chermareva's first, second, and third integral equations, respectively, for boundary conditions of the first, second, and third kind at $x = 0$.

We use t_0, $x_0 = at_0^2$, T_0 to denote the characteristic time, coordinate, and temperature (a is the thermal diffusivity of the medium) and we introduce the dimensionless time $\tau = t/t_0$, the phase front coordinate $y(\tau) = \xi(t)/x_0$, and the temperature of the boundary $u_0(\tau) = T(0, \tau)/T_0$. In terms of these variables it is convenient to write Grinberg and Chekmareva's first integral equation, bearing in mind further transformations in the form

$$\int_0^\infty \exp(-p\tau)\left\{\mathrm{ch}\,[p^{1/2} y(\tau)] - 1\right\} d\tau = 2\mu^2 \hat{u}_0(p). \tag{1}$$

Here $\hat{u}_0(p)$ is the Laplace transform of $u_0(\tau)$, and $\mu = (cT_0/2L)^{1/2}$ is a dimensionless parameter. The volumetric specific heat and the latent heat of melting will be denoted by c and L.

It is unlikely that nonlinear integral equation (1) has an exact solution for an arbitrary function $\hat{u}_0(p)$. However, when the condition $\mu \ll 1$ is satisfied, a solution of Eq. (1) can be found in the general case in the form of an asymptotic series in μ. It should be noted that in some cases it is possible to sum this series and thereby find an exact solution.

The condition $\mu \ll 1$ is equivalent to satisfaction of the condition of smallness of the volumetric energy $- cT_0$ of heating from the initial temperature to the maximum temperature T_0 relative to the volumetric latent heat of melting L.

2. To save space, we will use the notation $s = p^{1/2}$, $y \equiv y(\tau, \mu)$, $F \equiv \mathrm{F}(\tau, s, \mu) = \mathrm{ch}(sy) - 1$, and we seek the functions y and F as power series in μ, assuming that differentiation with respect to μ and integration with the weight factor $\exp(-p\tau)$ with respect to τ are valid for these series:

$$y = \sum_{k=1}^{\infty} \frac{\mu^k}{k!} y_k, \quad F = \sum_{k=2}^{\infty} \frac{\mu^k}{k!} F_k, \quad y_k \equiv y(\tau, s), \quad F_k \equiv F_k(\tau, s). \tag{2}$$

In writting these series, we took into account that $y(\tau, 0) \equiv F(\tau, s, 0) \equiv F^{(1)}(\tau, s, 0) \equiv 0$. Here and in the following the expression $v^{(k)}$ means the k-th derivative of the function v with respect to μ.

It is evident that

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$$y_k = y^{(k)}(\tau, 0), \quad F_k = F^{(k)}(\tau, s, 0). \tag{3}$$

3. It will be shown how $y_k(\tau)$ can be found successively for $k = 1, 2, \ldots$. To do this, the second of the series in formula (2) will be substituted in Eq. (1), and terms with equal powers of μ will be equated. This leads to an infinite system of integral equations for determination of $y_k(\tau)$:

$$\int_0^\infty \exp(-p\tau) F_2(\tau, s)\, d\tau = 4\hat{u}_0(p), \tag{4}$$

$$\int_0^\infty \exp(-p\tau) F_k(\tau, s)\, d\tau = 0, \quad k = 3, 4, \ldots \tag{5}$$

Now F_k will be expressed in terms of y_k. It is obvious that for $k \geq 1$ $F^{(k)}$ can be written as

$$F^{(k)} = f_k \operatorname{ch}(sy) + \psi_k \operatorname{sh}(sy), \quad f_k \equiv f_k(\tau, s, \mu),$$
$$\psi_k \equiv \psi_k(\tau, s, \mu), \quad f_1 \equiv 0, \quad \psi_1 = sy. \tag{6}$$

Therefore,

$$F_k = F^{(k)}(\tau, s, 0) = f_k(\tau, s, 0). \tag{7}$$

To determine f_k from recurrence formulas, expression (6) will be differentiated with respect to μ:

$$F^{(k+1)} = [f_k^{(1)} + sy^{(1)}\psi_k] \operatorname{ch}(sy) + [\psi_k^{(1)} + sy^{(1)} f_k] \operatorname{sh}(sy) =$$
$$= f_{k+1} \operatorname{ch}(sy) + \psi_{k+1} \operatorname{sh}(sy),$$

whence it follows that

$$f_{k+1} = f_k^{(1)} + sy^{(1)}\psi_k, \quad \psi_{k+1} = \psi_k^{(1)} + sy^{(1)} f_k. \tag{8}$$

From formulas (7) and (8) F_2, F_3, etc. can be easily found in succession.

The expressions for the three first values of F_k will be given, omitting simple calculations:

$$F_2 = py_1^2(\tau), \quad F_3 = 3py_1(\tau)\, y_2(\tau), \quad F_4 = p^2 y_1^4(\tau) + 4y_1(\tau)\, y_3(\tau). \tag{9}$$

Substitution of F_k from formula (9) into integral equations (4) and (5) yields

$$\int_0^\infty \exp(-p\tau)\, y_1^2(\tau)\, d\tau = \frac{4\hat{u}_0(p)}{p}, \tag{10}$$

$$\int_0^\infty \exp(-p\tau)\, y_1(\tau)\, y_2(\tau)\, d\tau = 0, \tag{11}$$

$$\int_0^\infty \exp(-p\tau)\, [py_1^4(\tau) \neq 4y_1(\tau)\, y_3(\tau)]\, d\tau = 0, \tag{12}$$

whence it is readily determined in succession that

$$y_1(\tau) = 2\left[\int_0^\tau u_0(\tau)\,d\tau\right]^{1/2}, \quad y_2(\tau) \equiv 0, \quad y_3(\tau) = -\frac{1}{3}\frac{d}{d\tau}[y_1(\tau)]^3. \tag{13}$$

Substitution of y_k from formula (13) into series (2) for $y(\tau)$ gives with accuracy to terms of fourth order in μ

$$y(\tau) = \mu\left[y_1(\tau) - \frac{1}{6}\mu^2 y_1^2(\tau)\,y_1(\tau)\right] + O(\mu^4). \tag{14}$$

4. To investigate the character of convergence of the series of $y(\tau)$ in μ, we will consider the well-known example of the exact solution of Stefan's problem for $u_0(\tau) \equiv 1$, which in the present notation has the form

$$y(\tau) = 2\beta\sqrt{\tau}, \tag{15}$$

where β is the root of the transcendental equation

$$\beta \exp\beta^2 \int_0^\beta \exp(-z^2)\,dz - \mu^2 = 0.$$

The derivative of the left-hand side with respect to β for $\beta = \mu = 0$ equals zero. Therefore β is not an analytical function of μ. However, for $\mu \ll 1$, it is possible to obtain an asymptotic expansion of β in μ. With accuracy to terms of order $O(\mu^4)$, we have

$$\beta = \mu - \frac{\mu^3}{3} + O(\mu^4),$$

whence, using formula (12), we obtain

$$y(\tau) = 2\mu\left(1 - \frac{\mu^2}{3}\right)\sqrt{\tau} + O(\mu^4). \tag{16}$$

On the other hand, it follows from formulas (13) that $y_1(\tau) = 2\tau^{1/2}$, $y_2(\tau) \equiv 0$, and $y_3(\tau) = -4\tau^{1/2}$. Substituting the values of y_1, y_2, y_3 into series (2) for $y(\tau)$, we obtain with accuracy to terms of order $O(\mu^4)$:

$$y(\tau) = 2\mu\left(1 - \frac{1}{3}\mu^2\right)\sqrt{\tau} + O(\mu^4). \tag{17}$$

The agreement between formulas (16) and (17) proves that the expansion of $y(\tau)$ in μ is an asymptotic series.

In conclusion, it should be noted that the method suggested for solution of nonlinear equation (1) in the case of the small parameter μ can easily be extended to the case where $\hat{u}_0(p)$ depends on μ in such a way that the function $\hat{u}_0(p)$ could be expanded in an analytical or asymptotical series in μ.

REFERENCES

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